

Remarks on Conserved Quantities and Entropy of BTZ Black Hole Solutions. Part II: BCEA Theory

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Abstract. The BTZ black hole solution for $(2 + 1)$ -spacetime is considered as a solution of a triad-affine theory (BCEA) in which topological matter is introduced to replace the cosmological constant in the model. Conserved quantities and entropy are calculated via Nöther theorem, reproducing in a geometrical and global framework earlier results found in the literature using local formalisms. Ambiguities in global definitions of conserved quantities are considered in detail.

A dual and covariant Legendre transformation is performed to re-formulate BCEA theory as a purely metric (natural) theory (BCG) coupled to topological matter. No ambiguities in the definition of mass and angular momentum arise in BCG theory. Moreover, gravitational and matter contributions to conserved quantities and entropy are isolated. Finally, a comparison of BCEA and BCG theories is carried out by relying on the results obtained in both theories.

1. Introduction

This paper is the continuation of [1], which is hereafter referred to as Part I. Notation follows closely.

In Part I we applied a geometrical, global and general formalism to study conserved quantities of BTZ black hole solution (see [2], [3], [4], [5], [6] and [7]) in $(2+1)$ dimensions. These particular solutions are realistic enough to be extensively used as test models for various approaches to conserved quantities and entropy in General Relativity. We hereafter consider so-called BCEA theory (see [7], [6]) which admits a BTZ black hole solution in a frame-affine formalism, as well as the corresponding purely metric theory, which will be called BCG theory. The BCG theory is obtained from BCEA theory by means of a “dual” covariant Legendre transformation. This is an application of a fairly general framework which applies to a much wider class of covariant theories (see for example [8], [9], [10], [11], [12], [13] and references quoted therein) and which is in fact the theoretical basis to explain also the meaning of the well known alternative variables for gravity known as *Ashtekar’s variables* (see [8], [10], [11], [12]). The main feature of BCG theory is that it canonically separates the contributions to conserved quantities and entropy of gravitational and “matter” parts. Various contributions have been *guessed* in literature (see [6]) but, as we shall show hereafter, the separation cannot be done in BCEA theory where the various contributions are tangled up.

Let us first recall the main results of Part I which shall be useful in the sequel. We start from the standard Hilbert-Einstein Lagrangian with cosmological constant

$$L = \mathcal{L} \, \mathbf{ds} = \alpha(r - 2\Lambda) \sqrt{g} \, \mathbf{ds} \quad (1.1)$$

where $\alpha \neq 0$ is a constant and notice that the BTZ metric (see [2], [3], [4], [5])

$$g_{\text{BTZ}} = -N^2 dt^2 + N^{-2} dr^2 + r^2 (N_\phi dt + d\phi)^2 \quad (1.2)$$

is a solution of Einstein field equations, where we set

$$\begin{aligned} N^2 &= -\mu + \frac{r^2}{l^2} + \frac{J^2}{4r^2} & \mu &= \frac{r_+^2 + r_-^2}{l^2} \\ N_\phi &= -\frac{J}{2r^2} & J &= 2\frac{r_+ r_-}{l} \end{aligned} \quad (1.3)$$

Then we apply the variational prescription for Nöther theorem (see Part I) and obtain for a generic conserved quantity $\hat{Q}(L, \xi, g)$ the following recipe

$$\delta_X \hat{Q}(L, \xi, g) = \int_\infty \left(\delta_X \mathcal{U}(L, \xi, g) - i_\xi (\mathcal{F}(L, g)[X]) \right) \quad (1.4)$$

where: X is any deformation of configuration fields; δ_X denotes variation along X ; ∞ is the asymptotic boundary of spacetime (i.e. space infinity); ξ is a vector field over spacetime M ; g is any Lorentzian metric over M ; $\mathcal{U}(L, \xi, g)$ is the *superpotential* of the Lagrangian (1.1), given by equation (3.7) in Part I; $\mathbb{F}(L, g)[X]$ is the map, called *Poincaré-Cartan morphism*, given by equation (4.9) in Part I. For conserved quantities of BTZ solutions we find

$$\hat{Q}_D(L, \partial_t, g_{\text{BTZ}}) = 2\pi\alpha\mu \quad \hat{Q}_D(L, \partial_\phi, g_{\text{BTZ}}) = -2\pi\alpha J \quad (1.5)$$

associated to symmetry generators ∂_t and ∂_ϕ , respectively. By applying a suitable covariant ADM formalism one can obtain the same results by means of the following expression

$$\hat{Q}_D(L, \xi, g) = \int_{\partial D} [\mathcal{U}(L, \xi, g) - \tilde{\mathcal{B}}(L, \xi, g)] \quad (1.6)$$

Here the correction $\tilde{\mathcal{B}}(L, \xi, g)$ is defined by pull-back along a section g of the following quantity

$$\tilde{\mathcal{B}}(L, \xi) = [p^{\mu\nu} w_{\mu\nu}^\lambda \xi^\rho + \alpha \sqrt{h} h^{\alpha\rho} \overset{(h)}{\nabla}_\alpha \xi^\lambda] \mathbf{ds}_{\lambda\rho} \quad (1.7)$$

where $p^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial r_{\mu\nu}}$ are the *covariant naive momenta*, $\gamma_{\mu\nu}^\lambda$ denotes the Levi-Civita connection of g , $h = \|h_{\mu\nu}\|$ is any reference *background solution*, $\Gamma_{\mu\nu}^\lambda$ denotes the Levi-Civita connection of h and we set

$$\begin{aligned} w_{\mu\nu}^\lambda &= u_{\mu\nu}^\lambda - U_{\mu\nu}^\lambda & u_{\mu\nu}^\lambda &:= \gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \gamma_{\nu)}^\epsilon \epsilon \\ U_{\mu\nu}^\lambda &:= \Gamma_{\mu\nu}^\lambda - \delta_{(\mu}^\lambda \Gamma_{\nu)}^\epsilon \epsilon \end{aligned} \quad (1.8)$$

We also recall that the density integrated in (1.6) can be obtained as the superpotential of the following first order Lagrangian

$$\tilde{L}_1 = [\alpha(r - 2\Lambda)\sqrt{g} - d_\lambda(p^{\mu\nu} w_{\mu\nu}^\lambda) - \alpha(R - 2\Lambda)\sqrt{h}] \mathbf{ds} \quad (1.9)$$

where d_λ is the formal divergence operator defined in Part I and R is the scalar curvature of the background h . For what concerns entropy we recall that there is a general prescription for the variation of the entropy

$$\delta_X \mathcal{S} = \frac{1}{T} \int_\Sigma \left(\delta_X \mathcal{U}(L, \xi, g) - i_\xi(\mathbb{F}(L, g)[X]) \right) \quad (1.10)$$

where $\xi = \partial_t + \Omega \partial_\phi$, Ω is the *angular velocity of horizon*, T is the temperature of the black hole and Σ is a $(n - 2)$ -submanifold such that $\infty - \Sigma$ is a homological boundary (i.e. it does not enclose any singularity). Equation (1.10) is equivalent to the first principle of thermodynamics and it gives, for any BTZ solution, the following result

$$\mathcal{S} = 8\pi^2 \alpha r_+ \quad (1.11)$$

We shall reproduce and compare these results in BCEA framework.

2. BCEA theories

Let us fix a $\text{SO}(2, 1)$ -principal bundle $\mathcal{F} = (F, M, \pi, \text{SO}(2, 1))$ over a 3-dimensional spacetime manifold M . Following [14] we define a *moving frame* to be any principal vertical morphism $e : \mathcal{F} \rightarrow L(M)$ into the linear frame bundle of M , i.e.

$$\begin{array}{ccc} F & \xrightarrow{e} & L(M) \\ \pi \downarrow & & \downarrow \\ M & \xrightarrow{id} & M \end{array} \quad \begin{array}{ccc} F & \xrightarrow{e} & L(M) \\ \tilde{R}_a \downarrow & & \downarrow R_a \\ F & \xrightarrow{e} & L(M) \end{array} \quad \forall a \in \text{SO}(2, 1) \subset \text{GL}(3)$$

where \tilde{R}_a and R_a denote the relevant canonical right actions of a on principal bundles.

Of course, depending on the choice of M and \mathcal{F} , moving frames may or may not exist (see [14]). We stress once more that by (*global*) *moving frame* we mean a global morphism $e : F \rightarrow L(M)$, the existence of which does not require the existence of a global section in $L(M)$: in other words, (*global*) *moving frames* may exist even on non parallelizable manifolds. However, if M allows a metric g of signature $\eta = (2, 1)$ it is always possible to choose a bundle \mathcal{F} so that global moving frames exist; as in [14] we call such bundles *structure bundles*. In the sequel, \mathcal{F} will be assumed to be a structure bundle. Although we are not directly interested here in bundle-theoretic aspects, we remark that there exists a bundle associated to \mathcal{F} , whose global sections corresponds to global moving frames (see [14], [15] for greater details).

Let us choose trivializations ς in \mathcal{F} and ∂_μ in $L(M)$, considered as a family of local sections as it is standard when dealing with principal fiber bundles. Using induced fibered coordinates, one can write a moving frame as

$$e : \varsigma \mapsto e_a \equiv e_a^\mu \partial_\mu \quad (2.1)$$

so that e_a^μ form an invertible matrix and they may be regarded as the *components* of the moving frame e ; e_a ($a = 1, 2, 3 = \dim(M)$) denote a linear

frame over M , i.e. a basis in its tangent space. Let us consider a principal automorphism in the structure bundle \mathcal{F} , which is locally given by

$$\begin{cases} x' = f(x) \\ a' = \alpha(x) \cdot a \end{cases} \quad \alpha(x) \in \text{SO}(2, 1) \quad (2.2)$$

where (x, a) are local fibered coordinates on \mathcal{F} . This automorphism acts on moving frames as follows

$$(e')^\mu_a = J^\mu_\nu e^\nu_b \bar{\alpha}^b_a \quad (2.3)$$

where $\|\bar{\alpha}^b_a\|$ is the inverse matrix of $\alpha(x)$ and $J^\mu_\nu = \partial_\nu f^\mu$ is the Jacobian of the diffeomorphism f induced on spacetime M .

Let e^μ_μ denote the inverse matrix of e^μ_a . Any moving frame e induces a metric $g_{\mu\nu} = e^\mu_\mu \eta_{ab} e^\nu_b$ where η_{ab} is the standard diagonal matrix of signature $\eta = (2, 1)$; g is defined so that its orthonormal frame bundle $\text{SO}(M, g)$ (which is the sub-bundle of $L(M)$ of g -orthonormal frames) coincides with the image of the moving frame e . We understand that Latin indices are raised and lowered by η_{ab} , while Greek indices are moved by the induced metric $g_{\mu\nu}$. Notice that because of $e^{a\mu} = g^{\mu\nu} e^\nu_a = \eta^{ab} e^\mu_b$ no possible confusion arises. We shall denote by A^{ab}_μ any principal $\text{SO}(2, 1)$ -connection of \mathcal{F} (not necessarily induced by the frame). In dimension three one can use the parametrization for $\text{SO}(2, 1)$ -connections given by:

$$A^a_\mu = \epsilon^a_{bc} A^{bc}_\mu \quad (2.4)$$

which is better suited for a comparison with the current literature (see [6]). In this notation the structure constants of $\text{SO}(2, 1)$ are $c^a_{bc} = \frac{1}{2} \epsilon^a_{bc}$.

Following [6] and [7], let us also introduce the “matter” fields b^a_μ and c^a_μ whose nature is analogous to e^μ_μ , though they are allowed to be degenerate. Let us consider the $\text{SO}(2, 1)$ -connection A^{ab}_μ and the Levi-Civita connection $\gamma^\lambda_{\sigma\mu}$ induced by the moving frame e through the induced metric $g_{\mu\nu}$; we can then define the covariant derivative of the moving co-frame e^μ_μ as:

$$\begin{aligned} \nabla^{(A)}_\mu e^a_\nu &= d_\mu e^a_\nu + A^{a\cdot}_{b\mu} e^b_\nu - \gamma^\lambda_{\nu\mu} e^a_\lambda \\ \nabla^{(A)} e^a &= \nabla^{(A)}_\mu e^a_\nu dx^\mu \wedge dx^\nu = (d_\mu e^a_\nu + A^{a\cdot}_{b\mu} e^b_\nu) dx^\mu \wedge dx^\nu \end{aligned} \quad (2.5)$$

and analogously for the fields b^a_μ and c^a_μ .

Consider now a field theory for the configuration fields $(e_\mu^\mu, A_\mu^a, b_\mu^a, c_\mu^a)$. Let us introduce the field strength 2-form of the $\text{SO}(2, 1)$ -connection A_μ^a

$$F^a = \frac{1}{2} F_{\mu\nu}^a dx^\mu \wedge dx^\nu = \frac{1}{2} (d_\mu A_\nu^a - d_\nu A_\mu^a + \frac{1}{2} \epsilon_{bc}^{a \cdot \cdot} A_\mu^b A_\nu^c) dx^\mu \wedge dx^\nu \quad (2.6)$$

or equivalently of A_μ^{ab}

$$F^{ab} = \frac{1}{2} F_{\mu\nu}^{ab} dx^\mu \wedge dx^\nu = (d_\mu A_\nu^{ab} + A_{c\mu}^a A_\nu^{cb}) dx^\mu \wedge dx^\nu \quad (2.7)$$

One can easily obtain the relation $F_{\mu\nu}^a = \epsilon_{bc}^{a \cdot \cdot} F_{\mu\nu}^{bc}$.

We can define the BCEA Lagrangian (see [7], [6]) by:

$$\begin{aligned} L_{BCEA} &= \mathcal{L}_{BCEA} \mathbf{ds} = -\alpha \eta_{ab} \left(\frac{1}{2} e_\mu^a F_{\nu\rho}^b + b_\mu^a \nabla_\nu^{(A)} c_\rho^b \right) \epsilon^{\mu\nu\rho} \mathbf{ds} = \\ &= \mathcal{L}_{EA} \mathbf{ds} + \mathcal{L}_{BC} \mathbf{ds} = \left(-\alpha \eta_{ab} e^a \wedge F^b \right) + \left(-\alpha \eta_{ab} b^a \wedge \nabla^{(A)} c^b \right) \end{aligned} \quad (2.8)$$

This is a kind of gauge theory with $\text{SO}(2, 1)$ as structure group. From the Lagrangian (2.8) one obtains the following field equations:

$$\begin{aligned} F^a &= 0 \\ \nabla^{(A)} e^a + \frac{1}{2} \epsilon_{bc}^{a \cdot \cdot} b^b \wedge c^c &= 0 \\ \nabla^{(A)} b^a &= 0 \\ \nabla^{(A)} c^a &= 0 \end{aligned} \quad (2.9)$$

A possible solution for these equations, in the gauge $A_\mu^a = 0$, is (see [7], [6]):

$$e = \begin{vmatrix} \frac{\lambda r_+}{l} & 0 & -\lambda r_- \\ 0 & \frac{l}{\nu} \frac{\partial}{\partial r}(\lambda) & 0 \\ -2\frac{\nu r_+}{l^2} & 0 & 2\frac{\nu r_-}{l} \end{vmatrix} \quad \begin{cases} \nu^2 = \frac{r_-^2 - r_+^2}{r_+^2 - r_-^2} \\ \lambda^2 = \frac{r_-^2 - r_+^2}{r_+^2 - r_-^2} \end{cases} \quad (2.10)$$

$$b = \begin{vmatrix} \frac{r_-}{l} & 0 & -r_+ \\ 0 & -l \frac{\partial}{\partial r}(\nu + \lambda) & 0 \\ \frac{r_+}{l} & 0 & -r_- \end{vmatrix} \quad c = \begin{vmatrix} -\frac{r_-}{l^2} & 0 & \frac{r_+}{l} \\ 0 & \frac{\partial}{\partial r}(\nu - \lambda) & 0 \\ \frac{r_+}{l^2} & 0 & -\frac{r_-}{l} \end{vmatrix} \quad (2.11)$$

where r_+ and r_- are the outer and inner horizons of a BTZ black hole.

Notice that the metric induced by the moving frame (2.10) coincides in fact with the BTZ solution (1.2). In some sense, this particular choice for the topological matter term due to fields (b, c) *simulates* the negative cosmological constant of the Lagrangian (1.1). However, we stress that these two theories are *not* equivalent even if for any solution of the first theory there is a solution of the BCEA theory which induces the same metric, so that in a suitable sense they both describe the same spacetimes (see [6]). In BCEA theories one has indeed additional fields (b^a, c^a) which, at least in principle, are expected to contribute to mass and angular momentum. Moreover, from a mathematical viewpoint the structure of their symmetry groups is completely different. The Lagrangian (1.1) is covariant with respect to diffeomorphisms of spacetime, i.e. it describes a *natural theory* owing to the fact that it is a purely metric theory. The BCEA theory is covariant with respect to $\text{Aut}(\mathcal{F})$; locally, this amounts to require covariance with respect to both diffeomorphisms of spacetime and local (pseudo)-rotations. Globally, this splitting of $\text{Aut}(\mathcal{F})$ is meaningless since there is no natural way of defining an action of diffeomorphisms alone on a moving frame. [Moving frames are basically a family of local linear frames whose transition functions are $\text{SO}(\eta)$ -valued; if we let a spacetime diffeomorphism to act on the family then we still get a family of frames but transition functions are no longer $\text{SO}(\eta)$ -valued. Thus the transformed family of local frames do not define in general a global metric with the correct signature on spacetime.] Having set up a geometric framework using fiber bundles together with their automorphisms, we are able to develop a global theory which does not use the local splitting of the symmetry group $\text{Aut}(\mathcal{F})$. Thus from a local viewpoint we have two (inequivalent) local theories, but from a global viewpoint just one of them is meaningful.

As we fixed the trivialization ς of $\mathcal{F} = (F, M, \pi, \text{SO}(2, 1))$, any point $p \in F$ (with $\pi(p) = x \in M$) may be written as $p = R_\theta \varsigma(x)$, where $\theta = \|\theta_b^a\| \in \text{SO}(2, 1)$; then we can define a pointwise basis of vertical right-invariant vector fields on \mathcal{F} :

$$\sigma_{ab} = \frac{1}{2}(\eta_{ac} \rho_b^c - \eta_{bc} \rho_a^c) \quad \rho_b^a = \theta_c^a \frac{\partial}{\partial \theta_c^b} \quad (2.12)$$

Now, let us consider an infinitesimal symmetry generator:

$$\Xi = \xi^\mu(x) \partial_\mu + \xi^{ab}(x) \sigma_{ab} \quad (2.13)$$

which is a right-invariant (thence projectable) vector field over the structure bundle \mathcal{F} . Using the action (2.3) of $\text{Aut}(\mathcal{F})$ on configuration fields we can

define by standard techniques (see [16]) the Lie derivatives of fields with respect to Ξ obtaining:

$$\begin{aligned}
\mathcal{L}_{\Xi} e_{\mu}^a &= \nabla_{\nu}^{(A)} e_{\mu}^a \xi^{\nu} + e_{\nu}^a \nabla_{\mu} \xi^{\nu} - e_{b\mu} \xi_{(V)}^{ab} \\
\mathcal{L}_{\Xi} A_{\mu}^a &= F_{\nu\mu}^a \xi^{\nu} + \epsilon_{bc}^{a\cdot\cdot} \nabla_{\mu}^{(A)} \xi_{(V)}^{bc} \\
\mathcal{L}_{\Xi} b_{\mu}^a &= \nabla_{\nu}^{(A)} b_{\mu}^a \xi^{\nu} + b_{\nu}^a \nabla_{\mu} \xi^{\nu} - b_{b\mu} \xi_{(V)}^{ab} \\
\mathcal{L}_{\Xi} c_{\mu}^a &= \nabla_{\nu}^{(A)} c_{\mu}^a \xi^{\nu} + c_{\nu}^a \nabla_{\mu} \xi^{\nu} - b_{b\mu} \xi_{(V)}^{ab}
\end{aligned} \tag{2.14}$$

where we have set $\xi_{(V)}^{ab} = \xi^{ab} + A_{\mu}^{ab} \xi^{\mu}$ for the vertical part of Ξ with respect to the connection A_{μ}^{ab} (see [15]).

By considering the variation of the Lagrangian (2.8) and integrating by parts, we obtain a term related to field equations (2.9) plus the divergence of the following 2-form:

$$\begin{aligned}
\mathbb{F}(L_{BCEA}, g)[\delta A, \delta c] &= \mathbb{F}_{EA}[\delta A] + \mathbb{F}_{BC}[\delta c] = \\
&= (\alpha \eta_{ab} b^a \wedge \delta c^b) + (\alpha \eta_{ab} e^a \wedge \delta A^b)
\end{aligned} \tag{2.15}$$

which is the *Poincaré-Cartan morphism* for the BCEA theory.

Then, owing to $\text{Aut}(\mathcal{F})$ -covariance and Nöther's theorem, the Lagrangian (2.8) admits the following superpotential 1-form

$$\begin{aligned}
\mathcal{U}(L_{BCEA}, \Xi) &= \mathcal{U}_{EA}[\Xi] + \mathcal{U}_{BC}[\Xi] = \\
&= (\alpha \xi_{(V)}^{ab} \epsilon_{abc} e^c) + (-\alpha \xi^{\nu} c_{a\nu} b^a)
\end{aligned} \tag{2.16}$$

where Ξ is given by (2.13). Thus, we can define the variation of conserved quantities as

$$\delta \hat{Q}_D(L_{BCEA}, \Xi, g) = \int_{\partial D} \left(\delta \mathcal{U}(L_{BCEA}, \Xi, g) - i_{\xi} \mathbb{F}(L_{BCEA}, g)[\delta A, \delta c] \right) \tag{2.17}$$

where $\xi = \xi^{\mu} \partial_{\mu}$ is the projection of the symmetry generator Ξ on spacetime M .

We again remark that the BCEA theory is covariant with respect to $\text{Aut}(\mathcal{F})$ and there is no natural action of spacetime diffeomorphisms on configuration fields. [This is basically due to the fact that the local action of diffeomorphisms and pure gauge transformations do not commute.] Thence in previous formulae the infinitesimal generators ξ^{μ} and ξ^{ab} are so far completely unrelated. If

we want to compare conserved quantities along the solution (2.10) and (2.11) with purely metric theories, which are $\text{Diff}(M)$ -covariant, we need to establish some relation among infinitesimal symmetry generators.

3. Kosmann lift

As previously remarked (see also [15] and references quoted therein), despite there is no natural action of $\text{Diff}(M)$ on the configuration bundle, one can define a global action of *infinitesimal* symmetry generators. In geometrically oriented literature this is known as the *Kosmann lift of vector fields*. Basically, one starts with a vector field $\xi = \xi^\mu \partial_\mu$ on spacetime M and lifts it by means of the natural lift to the frame bundle $L(M)$ defined by

$$\hat{\xi} = \xi^\mu(x) \partial_\mu + d_\mu \xi^\nu(x) \rho_\nu^\mu \quad \rho_\nu^\mu = V_a^\mu \frac{\partial}{\partial V_a^\nu} \quad (3.1)$$

where (x^μ, V_a^μ) are fibered coordinates on $L(M)$ so that ρ_ν^μ is a pointwise basis for vertical right-invariant vector fields.

Now, by fixing a moving frame $e : \mathcal{F} \rightarrow L(M)$ one would somehow pull-back $\hat{\xi}$ over \mathcal{F} . Unfortunately, the pull-back of vector fields can be performed only along bijective maps, while here e , though injective, is not surjective. Nevertheless, if $\hat{\xi}$ were tangent to the image of \mathcal{F} through the moving frame e , its pull-back would be well defined. Thus, if we define a global way of projecting $\hat{\xi}$ to the tangent space of $\text{Im}(e)$ we can use it to define a global (though non natural) lift of vector fields over M to vector fields over \mathcal{F} . This lift depends on the moving frame e we chose and it establishes a relation between the generators ξ^μ and ξ^{ab} of infinitesimal symmetries.

Once we choose a moving frame e , the $\text{SO}(2,1)$ -connection ω_μ^{ab} , called the *spin connection*, is induced. It is *compatible* with the frame, i.e.

$$\overset{(\omega)}{\nabla}_\mu e_a^\nu = d_\mu e_a^\nu + \gamma_{\lambda\mu}^\nu e_a^\lambda - \omega_{a\mu}^b e_b^\nu = 0 \quad (3.2)$$

or, equivalently, it is related to the Levi-Civita connection $\gamma_{\mu\nu}^\lambda$ of the metric $g_{\mu\nu}$ induced by the frame e_a^μ itself as follows:

$$\omega_\rho^{ab} = e_\lambda^a (\gamma_{\mu\rho}^\lambda e^{b\mu} + d_\rho e^{b\lambda}) \quad (3.3)$$

Let (x^μ, θ_b^a) be the fibered coordinates induced in $L(M)$ by the trivialization ς fixed on \mathcal{F} and the moving frame e ; these *adapted coordinates* induce a pointwise basis for vertical right invariant vector fields:

$$\rho_b^a = \theta_c^a \frac{\partial}{\partial \theta_c^b} \quad (3.4)$$

We can recast the natural lift (3.1) with respect to these adapted coordinates as $\hat{\xi} = \xi^\mu(x) \partial_\mu + \hat{\xi}_b^a \rho_a^b$ by setting

$$\hat{\xi}_b^a = e_b^\mu \overset{(\omega)}{\nabla}_\mu (e_\nu^a \xi^\nu) - \omega_{b\mu}^a \xi^\mu = e_\rho^a (e_b^\mu d_\mu \xi^\rho - \xi^\mu d_\mu e_b^\rho) \quad (3.5)$$

We can now define the Kosmann lift (see [17]) as

$$\hat{\xi}_{(K)} = \xi^\mu(x) \partial_\mu + \hat{\xi}^{ab}(x) \sigma_{ab} \quad \hat{\xi}^{ab} = \hat{\xi}_c^{[a} \eta^{b]c} \quad \sigma_{ab} = \eta_{c[a} \rho_{b]}^c \quad (3.6)$$

We remark that σ_{ab} is a pointwise basis for vertical right-invariant vector fields on the sub-bundle $\text{Im}(e)$; σ_{ab} also denote the induced vector fields on \mathcal{F} . The projection of vector fields of $L(M)$ over $\text{Im}(e)$ is thence made by skew-symmetrization. We stress that the components $\hat{\xi}^{ab} = \hat{\xi}_c^{[a} \eta^{b]c}$ of the Kosmann lift do not depend on any connection, as clearly shown by equation (3.5).

We also stress that in BCEA theories the Kosmann lift (3.6) is just one of various possibilities of establishing a relation among ξ^μ and ξ^{ab} . The relation among the infinitesimal symmetry generators ξ^μ and ξ^{ab} certainly requires a *global* lift, which possibly does not depend on fields other than configuration ones in order to preserve covariance. However, in BCEA theory there are two $\text{SO}(2,1)$ -connections (namely, ω_μ^{ab} and A_μ^{ab}) which thence enable us to define (at least) four well-defined global lifts, given by the following expressions:

$$\hat{\xi}_b^a = e_b^\mu \overset{(\omega)}{\nabla}_\mu (e_\nu^a \xi^\nu) - \omega_{b\mu}^a \xi^\mu \quad (3.7)$$

$$\hat{\xi}_b^a = e_b^\mu \overset{(A)}{\nabla}_\mu (e_\nu^a \xi^\nu) - \omega_{b\mu}^a \xi^\mu \quad (3.8)$$

$$\hat{\xi}_b^a = e_b^\mu \overset{(\omega)}{\nabla}_\mu (e_\nu^a \xi^\nu) - A_{b\mu}^a \xi^\mu \quad (3.9)$$

$$\hat{\xi}_b^a = e_b^\mu \overset{(A)}{\nabla}_\mu (e_\nu^a \xi^\nu) - A_{b\mu}^a \xi^\mu \quad (3.10)$$

We stress again that all these lifts are global (though not natural) and in principle no one of them is, *a priori*, better than the others.

4. Conserved quantities and entropy in BCEA theories

We are now in the position of calculating the variation of the conserved quantities (2.17) along the solution (2.10) and (2.11) associated to the diffeomorphism generators ∂_t and ∂_ϕ through the lifts (3.7) - (3.10). The contribution of \mathbb{F} identically vanishes, i.e.:

$$\int_{\partial D} i_{\partial_t} \mathbb{F}(L_{BCEA}, g)[\delta A, \delta c] = 0, \quad \int_{\partial D} i_{\partial_\phi} \mathbb{F}(L_{BCEA}, g)[\delta A, \delta c] = 0 \quad (4.1)$$

We remark that this result does not rely on the choice of the lift because the vertical part of the symmetry generator does not enter expression (2.15). Let us recall that the vertical part of a lifted symmetry generator $\hat{\xi}$ is

$$\xi_{(V)}^{ab} \sigma_{ab} = (\hat{\xi}^{ab} + A_\mu^{ab} \xi^\mu) \sigma_{ab} \quad (4.2)$$

where $\hat{\xi}^{ab}$ may be any one of the lifts defined by (3.7) - (3.10); we thence obtain respectively the following results

$$\delta \hat{Q}[\partial_t] = \alpha \frac{2\pi}{l} \delta J \quad \delta \hat{Q}[\partial_\phi] = -2\pi \alpha l \delta \mu \quad (4.3)$$

$$\delta \hat{Q}[\partial_t] = \alpha \frac{2\pi}{l} \delta J \quad \delta \hat{Q}[\partial_\phi] = 2\pi \alpha \left(\frac{1}{2} \delta J - l \delta \mu \right) \quad (4.4)$$

$$\delta \hat{Q}[\partial_t] = \alpha \frac{2\pi}{l} \delta J \quad \delta \hat{Q}[\partial_\phi] = -2\pi \alpha (\delta J + l \delta \mu) \quad (4.5)$$

$$\delta \hat{Q}[\partial_t] = \alpha \frac{2\pi}{l} \delta J \quad \delta \hat{Q}[\partial_\phi] = -2\pi \alpha \left(\frac{1}{2} \delta J + l \delta \mu \right) \quad (4.6)$$

where integrals are performed on 1-spheres S_r^1 of radius r . One can easily integrate these quantities to obtain $\hat{Q}[\partial_t]$ and $\hat{Q}[\partial_\phi]$. We stress that all these quantities are conserved, though of course at most one of them can be interpreted as the *mass* and the other as the *angular momentum*. Notice that the quantities (4.3) agree with the result found in [6] using a locally defined (and globally ill-behaved) action of spacetime diffeomorphisms on configuration fields; notice in particular the exchange between the mass and the angular momentum: here they are related to the original (global) Kosmann lift (3.6). In other words, angular momentum μ is not always generated by ∂_ϕ but rather by another symmetry generator λ' given respectively by

$$\lambda' = \partial_\phi \quad (4.7)$$

$$\lambda' = \partial_\phi - \frac{l}{2} \partial_t \quad (4.8)$$

$$\lambda' = \partial_\phi + l \partial_t \quad (4.9)$$

$$\lambda' = \partial_\phi + \frac{l}{2} \partial_t \quad (4.10)$$

As already stressed in [6], the total conserved quantities determined here differ from the total conserved quantities (1.5) determined for the metric theory (1.1). As we already noticed, despite both theories describe the same spacetimes, they are clearly dynamically inequivalent, a fact that reverberates on the difference of conserved quantities. Finally, as shown above, one could directly compute conserved quantities by fixing a background as in (1.6) or, equivalently, by using a first order Lagrangian analogous to (1.9), obtaining the same results. Again the background is generally needed to avoid divergences.

Now, we can consider the Killing vector $\xi = \partial_t + \Omega \partial_\phi$ to calculate the entropy (1.10) using the Kosmann lift (3.6); we obtain

$$\delta_X \mathcal{S} = 8\pi^2 \alpha \delta r_- \quad (4.11)$$

which clearly gives $\mathcal{S} = 8\pi^2 \alpha r_-$. Notice the exchange of the role of the inner and outer horizons in the expression of entropy if compared with (1.11). Once again a different result with respect to (1.11) is due to the inequivalence of the theories.

The entropy has been calculated in a geometric framework; this result corroborates the results already obtained in [6]. We remark that if one tries to use one of the other lifts (3.8), (3.9) or (3.10), the expression for the variation of the entropy so obtained turns out to be non-integrable. Similarly, non-integrable expressions are obtained using the spin connection ω_μ^{ab} instead of A_μ^{ab} to define the vertical projection (4.2), i.e. $\xi_{(V)}^{ab} = \hat{\xi}^{ab} + \omega_\mu^{ab} \xi^\mu$. It once again selects the original Kosmann lift as a kind of *canonical choice*; a further hint in favour of this choice will arise in next Section.

Finally, we remark that another possibility for the lift of vector fields over M to vector fields over \mathcal{F} is to use directly the dynamical connection A_μ^{ab} , as one does in gauge theories. In this case one puts

$$\xi^{ab} = -A_\mu^{ab} \xi^\mu \quad (4.12)$$

which obviously gives $\xi_{(V)}^{ab} = 0$. It can be easily shown that by applying this choice to our example one obtains the same conserved quantities as in (4.3) and thence the same entropy. Curiously enough, these two lifts are therefore completely equivalent in the example under investigation, though they are completely unrelated in general. A comparison between them should then be carried over in other examples.

5. Purely metric BCG theories

As we said, in BCEA theory two $\text{SO}(2, 1)$ -connections are involved, namely the dynamical connection A_μ^{ab} and the spin connection ω_μ^{ab} ; as we noticed in the previous Sections, the freedom we have in choosing between them causes ambiguities when dealing with conserved quantities.

To overcome these difficulties, we hereafter want to define a field theory (BCG) equivalent (on-shell) to BCEA theory introduced in Section 2, in which only tensorial objects appear. Through a partial (covariant) Legendre transformation the BCEA Lagrangian (2.8) will be rewritten into a purely metric form in which the gravitational field is described by means of a metric on spacetime

in place of a $\text{SO}(2,1)$ -connection together with a moving frame as it happens in BCEA theory.

In BCEA theory the connection ω_μ^{ab} is determined by the compatibility condition

$$\overset{(\omega)}{\nabla} e^a = 0 \quad (5.1)$$

while the connection A_μ^{ab} satisfies field equation (2.9) :

$$\overset{(A)}{\nabla} e^a + \frac{1}{2} \epsilon_{bc}^{a \cdot \cdot} b^b \wedge c^c = 0 \quad (5.2)$$

If the “matter” Lagrangian $L_{BC} = \mathcal{L}_{BC} \mathbf{ds}$ were not present (or more generally if it did not depend on the connection) then the field equation (5.2) would reduce to a compatibility condition for A_μ^{ab} which ensures that $A_\mu^{ab} = \omega_\mu^{ab}$ on-shell. When this is the case, according to the so-called Palatini method (also known as *first order variational method*, see [13] and references quoted therein), one would calculate the connection with respect to the moving frame e and its first order derivatives. Then by substituting the expression so obtained into the Lagrangian one can recast it into the standard Hilbert-Einstein second order Lagrangian (possibly coupled with matter fields).

The “matter” Lagrangian $L_{BC} = -\alpha \eta_{ab} b_\mu^a \nabla^{(A)}_\nu c_\rho^b \epsilon^{\mu\nu\rho} \mathbf{ds}$ actually depends on the connection A_μ^{ab} through the covariant derivative $\nabla^{(A)}_\nu c_\rho^b$. Since A_μ^{ab} has to satisfy equation (5.2), it is no longer equal to the (compatible) spin connection ω_μ^{ab} . Nevertheless, since the matrix e_μ^a associated to the moving (co)frame is invertible, equation (5.2) can anyway be solved with respect to the equivalent connection A_μ^a (see (2.4)), giving a (unique) function of the moving frame e , its first order derivatives, together with b and c fields. By an easy (though tedious) calculation we obtain in fact

$$A_\mu^a = \epsilon_{bc}^{a \cdot \cdot} \omega_\mu^{bc} + e_\gamma^a \left[\epsilon^{\gamma\sigma\nu} \epsilon_{\rho\lambda\mu} b_\sigma^\rho c_\nu^\lambda + \frac{1}{2} \delta_\mu^\gamma \left(\text{Tr}(b \cdot c) - \text{Tr}(b) \text{Tr}(c) \right) \right] \quad (5.3)$$

where Tr denotes the trace of matrices and where we set

$$b_\lambda^\rho = e_a^\rho b_\lambda^a \quad c_\lambda^\rho = e_a^\rho c_\lambda^a \quad (5.4)$$

The pull-back of the Lagrangian (2.8) through the equation (5.3) (roughly speaking, substituting (5.3) into the Lagrangian (2.8)) defines the purely metric BCG Lagrangian

$$L_{BCG} = L_H + L_{\text{Int}} + L_{\text{Div}} + L_0 \quad (5.5)$$

where we set

$$L_H = \alpha \sqrt{g} r \, \mathbf{ds} \quad (5.6)$$

$$L_{\text{Int}} = -\alpha g_{\gamma\lambda} \epsilon^{\mu\nu\rho} b_\mu^\gamma \nabla_\nu c_\rho^\lambda \, \mathbf{ds} \quad (5.7)$$

$$L_{\text{Div}} = \alpha d_\nu \left[g_{\gamma\lambda} \epsilon^{\gamma\rho\sigma} \left(b_\rho^\nu c_\sigma^\lambda - b_\rho^\lambda c_\sigma^\nu \right) \right] \mathbf{ds} \quad (5.8)$$

$$L_0 = \frac{\alpha \sqrt{g}}{8} \left[2 \left(\text{Tr}(b^2) \text{Tr}(c^2) - 2 \text{Tr}(b^2 \cdot c^2) + \text{Tr}(b \cdot c)^2 \right) + \right. \\ \left. - \left(\text{Tr}(b \cdot c) - \text{Tr}(b) \text{Tr}(c) \right)^2 \right] \mathbf{ds} \quad (5.9)$$

where, from now on, ∇ denotes the (metric) covariant derivative with respect to the Levi-Civita connection of the metric g . The BCG Lagrangian is (on-shell) *equivalent* to BCEA Lagrangian (2.8): one obtains, in fact, field equations of the BCG Lagrangian by substituting the expression (5.3) into the field equations (2.9) of the BCEA Lagrangian.

Geometrically, one can interpret the expression (5.3) as the definition of the *partial (covariant) dual Legendre map*

$$(e_\mu^a, d_\nu e_\mu^a; b_\mu^a, c_\mu^a) \mapsto (e_\mu^a, A_\mu^a(e, de; b, c); b_\mu^a, c_\mu^a) \quad (5.10)$$

In fact, following the multisymplectic approach to the Hamiltonian formulation of field theories (see [8], [12], [18] and references quoted therein) we consider A_μ^a as the configuration fields, while their *covariant momenta* are not the derivatives of the Lagrangian density with respect to time derivatives of the A_μ^a (as in standard $(3+1)$ Hamiltonian formulation) but rather the derivatives of the Lagrangian density with respect to *all “generalized velocities”* $F_{\mu\nu}^a$, namely

$$p_a^{\mu\nu} = \frac{\partial \mathcal{L}_{\text{BCEA}}}{\partial F_{\mu\nu}^a} = -\frac{\alpha}{2} e_{a\rho} \epsilon^{\rho\mu\nu} \quad (5.11)$$

Following this approach, the moving frames e are, in practice, the conjugate fields to A_μ^a . In this way, the BCEA Lagrangian may be considered as the analogous of the Helmholtz Lagrangian $L_H = p \dot{q} - H(p, q)$ in Classical Mechanics, where the mechanical analogy is established by the correspondence $e \rightarrow p$, $A \rightarrow q$ and $F \rightarrow \dot{q}$; the term $p \dot{q}$ in L_H corresponds to the part $L_{EA} = -\alpha \eta_{ab} e^a \wedge F^b$ of the BCEA Lagrangian and H to the part L_{BC} .

In Mechanics, provided the regularity condition $\det |\frac{\partial^2 H}{\partial p \partial p}| \neq 0$ is satisfied, one can (at least locally) solve the Hamilton equation

$$\dot{q} = \frac{\partial H}{\partial p}(q, p) \quad (5.12)$$

with respect to p by expressing it as a function $\tilde{p}(q, \dot{q})$. This defines the Legendre transformation

$$(q, \dot{q}) \rightarrow (q, \tilde{p}(q, \dot{q})) \quad (5.13)$$

which can be used to define the Lagrangian $L(q, \dot{q}) = \tilde{p} \dot{q} - H(q, \tilde{p})$. [This approach may be used to give, for example, purely affine formulations of General Relativity (see [8], [9], [10], [11], [12]). As far as BCEA theory is concerned, however, one cannot follow this route since the Hamilton equation (5.12) corresponds to the field equation $F^a = 0$ which cannot be solved with respect to e since it does not appear at all in the relevant expression!]

On the contrary, one can consider the other Hamilton equation

$$\dot{p} = -\frac{\partial H}{\partial q}(q, p) \quad (5.14)$$

Provided the regularity condition $\det |\frac{\partial^2 H}{\partial q \partial q}| \neq 0$ is satisfied, the equation (5.14) may be solved with respect to q by expressing it as a function $\tilde{q}(p, \dot{p})$. It then defines the *dual Legendre transformation*

$$(p, \dot{p}) \rightarrow (p, \tilde{q}(p, \dot{p})) \quad (5.15)$$

which is analogous to (5.10), being the Hamilton equation (5.14) analogous to field equation (5.2). Then it has been shown in general (see [8]) that the other set of Hamilton equations (5.12), once we substitute \tilde{q} , may be viewed as the Lagrange equations of a *dual Lagrangian* $L^*(p, \dot{p}, \ddot{p})$ which is linear in \ddot{p} . This dual Lagrangian is the mechanical analogous of the BCG Lagrangian we built (which is clearly linear in second order derivatives of the metric $g_{\mu\nu}$).

Now, let us calculate conserved quantities. First of all, we remark that, for any Lagrangian L , the quantity $\delta_X \mathcal{U}(L, \xi, g) - i_\xi \mathcal{F}(L, g)[X]$ is invariant with respect to the addition of total divergences to the Lagrangian. Thence it is not at all affected by the term (5.8) in the BCG Lagrangian. Secondly, being the term (5.9) a zeroth order Lagrangian, it affects field equations but it is irrelevant to both $\mathcal{U}(L_{BCG}, \xi, g)$ and $\mathcal{F}(L_{BCG}, g)$. Thus, despite the BCG Lagrangian (5.5) is quite complicated, the only terms in the Lagrangian which are relevant to conserved quantities are L_H and L_{Int} .

By easy calculations, one can find the following

$$\begin{aligned} \mathcal{U}(L_H, \xi) &= \alpha \sqrt{g} \nabla^\mu \xi^\nu \epsilon_{\nu\mu\lambda} dx^\lambda \\ \mathcal{U}(L_{Int}, \xi) &= \alpha \xi^\gamma \left[\text{Tr}(c) b_{\gamma\lambda}^\cdot + \text{Tr}(b) c_{\gamma\lambda}^\cdot - b_{\gamma\nu}^\cdot c_\lambda^\nu - c_{\gamma\nu}^\cdot b_\lambda^\nu - b_{\nu\lambda}^\cdot c_\gamma^\nu + \right. \\ &\quad \left. + \frac{1}{2} g_{\lambda\gamma} \left(\text{Tr}(b \cdot c) - \text{Tr}(b) \text{Tr}(c) \right) \right] dx^\lambda \\ \mathcal{F}(L_H, g)[\delta g] &= \alpha (g^{\lambda\rho} g_{\mu\nu} - \delta_{(\mu}^\lambda \delta_{\nu)}^\rho) \nabla_\rho \delta g^{\mu\nu} \sqrt{g} ds_\lambda \\ \mathcal{F}(L_{Int}, g)[\delta g, \delta c] &= \alpha \left[\Delta^{\mu\gamma\nu} \delta g_{\gamma\nu} - \epsilon^{\mu\rho\sigma} b_{\lambda\sigma}^\cdot \delta c_\rho^\lambda \right] ds_\mu \end{aligned} \quad (5.16)$$

where we set

$$\Delta^{\mu\gamma\nu} = \frac{1}{2} \left[\epsilon^{\rho\nu\sigma} b_{\rho}^{\mu} c_{\sigma}^{\gamma} - \epsilon^{\rho\mu\sigma} b_{\rho}^{\gamma} c_{\sigma}^{\nu} - \epsilon^{\rho\gamma\sigma} b_{\rho}^{\nu} c_{\sigma}^{\mu} \right] \quad (5.17)$$

Replacing these expressions back into expression (1.4) we obtain the following variations of conserved quantities for BTZ solutions

$$\begin{aligned} \delta \hat{Q}(L_{BCG}, \partial_t, g) &= \alpha \frac{4\pi}{l^2} \left[r_- \delta r_+ + r_+ \delta r_- \right] \\ \delta \hat{Q}(L_{BCG}, \partial_{\phi}, g) &= -\alpha \frac{4\pi}{l} \left[r_+ \delta r_+ + r_- \delta r_- \right] \end{aligned} \quad (5.18)$$

which are readily integrated to

$$\hat{Q}(L_{BCG}, \partial_t, g) = \alpha \frac{2\pi}{l} J \quad \hat{Q}(L_{BCG}, \partial_{\phi}, g) = -2\pi \alpha l \mu \quad (5.19)$$

[We stress once again that the same result would be obtained by choosing as a background another BTZ solution with different parameters (r_+^0, r_-^0) and then using the correction (1.7).]

As a consequence, the variation of the entropy (see eq. (1.10)) has the value $\delta \mathcal{S} = 8\pi^2 \alpha \delta r_-$ so that \mathcal{S} is given by:

$$\mathcal{S} = 8\pi^2 \alpha r_- \quad (5.20)$$

We remark that the BCG theory is a natural theory since any dynamical field is a tensor over spacetime. Naturality means that there is a canonical action of spacetime diffeomorphisms over dynamical fields and no ambiguity arise about the lift to be used in Nöther theorem: a unique natural lift exists. Such a natural lift reproduces the results (4.3) and (4.11) and this again justifies *a posteriori* our choice of the Kosmann lift made in previous Sections.

We also remark that here the *minimal coupling* is physically meaningful, i.e. we can split the Lagrangian into a purely metric Lagrangian L_H , namely the Hilbert-Einstein Lagrangian, and “matter” and interaction Lagrangian $L_{\text{Int}} + L_{\text{Div}} + L_0$ not containig second derivatives of the metric field. The conserved quantities (5.19) and the entropy (5.20) receive thence two contributions, each one from each term of the splitting. These are

$$\begin{aligned} \hat{Q}_g \equiv \hat{Q}(L_H, \partial_t, g) &= 2\pi \alpha \mu & \hat{Q}_{BC} \equiv \hat{Q}(L_{\text{Int}}, \partial_t, g) &= 2\pi \alpha \left(\frac{1}{l} J - \mu \right) \\ \hat{Q}(L_H, \partial_{\phi}, g) &= -2\pi \alpha J & \hat{Q}(L_{\text{Int}}, \partial_{\phi}, g) &= 2\pi \alpha (J + l\mu) \\ \mathcal{S}_g &= 8\pi^2 \alpha r_+ & \mathcal{S}_{BC} &= 8\pi^2 \alpha (r_- - r_+) \end{aligned} \quad (5.21)$$

These results thoroughly justify the suggested interpretation of the exchange of inner and outer horizon in (4.11) as the resulting contribution of b^a and c^a fields to total entropy (see [6]). They also show how the same idea cannot be applied to BCEA theory where, because of the expression (5.3), it is impossible to isolate in the BCEA Lagrangian a purely gravitational part analogous to L_H . Roughly speaking, the Lagrangian L_{EA} , when rewritten as a metric Lagrangian by using (5.3), does not correspond to the Hilbert-Einstein vacuum Lagrangian L_H in (5.5).

6. Conclusion and Perspectives

We introduced and analysed BCEA and BCG theories. Conserved quantities have been calculated for BCEA theories but, since there is an additional gauge invariance, additional conserved quantities have arisen. They are related to pure gauge transformations (i.e. local (pseudo)-rotations). Moreover, since the symmetry group of the theory does not naturally split into a vertical part (related to pure gauge transformations) and a horizontal part (related to diffeomorphisms of spacetime), the problem of a global definition of mass and angular momentum arises. We have faced this problem by considering various lifts which, although not canonical, are still globally well-defined. The so-called Kosmann lift was introduced in a different context (see [17]) to justify a proposal due to Kosmann for the Lie derivative of spinor fields with respect to general (i.e. not necessarily Killing) vector fields tangent to spacetime. The generalizations of the Kosmann lift here considered were suggested by the fact that in BCEA theory there are two dynamical connections, namely A_μ^{ab} and ω_μ^{ab} .

The lifts we considered usually produce different quantities, which, even if conserved, are not directly the expected *mass* and *angular momentum* of a BTZ black hole solution, but a combination of the two, owing to a pure gauge contribution. Two of them, i.e. the Kosmann lift and a suitably defined *gauge lift* produced conserved quantities which both agree with values predicted by BCG theory, where there is no ambiguity in the definition of conserved quantities, being the latter a natural (generally covariant) theory. This result suggests that these two lifts have to be somehow preferred with respect to the other possible lifts considered. This claim is also supported by the fact that the equation obtained for the variation of the entropy is integrable only for these two lifts, and the entropy computed again agrees with the BCG result. However, we stress that both lifts (the Kosmann and the gauge one) are well-defined: they do not depend on non-dynamical background fields, so that they are completely determined in terms of the solution under consideration, and moreover they both produce the same conserved quantities. As far as we know there is no

reason, other than aesthetical, to choose between them when considering a BTZ solution. Thence future investigations will be addressed to study other solutions in which the two lifts might give different predictions which could possibly help in selecting a *better* one from good standing physical and/or mathematical grounds.

The BCG theory is defined starting from BCEA theory and performing a dual covariant Legendre transformation. We stress that the so-called Palatini (or first order) variational method (see [13]) is a particular case of the Legendre transformation we performed. Palatini method correctly applies to frame-affine theories when the coupling to matter fields does not depend on derivatives of the moving frame e . Using the mechanical analogy we introduced, one could say that, if a Lagrangian $L = \frac{m}{2}(\dot{q})^2 - U(q)$ is considered, Palatini method corresponds to define momenta as $p = m\dot{q}$ while for more general Lagrangians $L = \frac{m}{2}(\dot{q})^2 - U(q, \dot{q})$ a contribution is due to the generalized potential, i.e. $p = \frac{\partial L}{\partial \dot{q}} = m\dot{q} - \frac{\partial U}{\partial \dot{q}}$. In BCEA Lagrangian, in fact, the coupling is defined through covariant derivatives, so that it depends on the dynamical connection A_μ^a : an extra contribution $b \wedge c$ arises so that the dynamical connection A_μ^{ab} does not coincide with the spin connection ω_μ^{ab} .

Furthermore, extra degrees of freedom owned by the moving frame (i.e. those related to local rotations) disappear when performing the Legendre transformation. This is an effect of the covariant choice of momenta. Thence BCG theory is a natural theory, i.e. it is generally covariant with respect to diffeomorphisms of spacetime. Contrarily to BCEA theory, in this theory there is a well defined action of diffeomorphisms on dynamical fields, due to their tensorial character. Derivation of mass and angular momentum is thence a standard application of a general framework (see [19] and references quoted therein) and no ambiguities are involved. Thus, one could say that BCG theory provided the expected values that BCEA theory should reproduce by constraining the choice of the lift.

Future investigations will be devoted to study the general mechanism which is behind the loss of naturality when passing from a purely metric to a frame-affine (or purely affine) formalism (see [8], [20], [21], [22], [23]). In these cases some sort of lift has always to be defined to replace the natural lift one has in the natural purely metric formalism, though no general prescription is known to be *a priori* correct. Furthermore, the BTZ solution has provided us a good test model for studying the transformation laws of conserved quantities (and entropy) under Legendre transformations. Other examples have been already investigated in literature (see [24] and references quoted therein) in a different framework. We claim that this subject deserves a geometrical setting in a more general theoretic context, which is currently under consideration and will form the subject of forthcoming papers.

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8. References

- [1] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, *Remarks on Conserved Quantities and Entropy of BTZ Black Hole Solutions. Part I: The General Setting*, (submitted)
- [2] M. Bañados, C. Teitelboim, J. Zanelli, Phys. Rev. Lett. **69**, 1849 (1992)
- [3] J. D. Brown, J. Creighton, R. B. Mann, Phys. Rev. D**50**, 6394 (1994)
- [4] D. Cangemi, M. Leblanc, R. B. Mann, Phys. Rev. D**48**, 3606 (1993)
- [5] S. Carlip, C. Teitelboim, Phys. Rev. D**51** (2), 622 (1995)
- [6] S. Carlip, J. Gegenberg, R. B. Mann, Phys. Rev. D**51** (12), 6854 (1995)
- [7] S. Carlip, J. Gegenberg, Phys. Rev. D**2** (44), 424 (1991)
- [8] M. Ferraris, M. Francaviglia, M. Raiteri, Quaderni del Dipartimento di Matematica **35** (1998) (submitted to J. Math. Phys.)
- [9] R. Capovilla, T. Jacobson, J. Dell, Phys. Rev. Lett. **63**, 2325 (1989)
- [10] R. Capovilla, T. Jacobson, J. Dell, L. Mason, Class. Quantum. Grav. **8**, 41 (1991)
- [11] R. Capovilla, T. Jacobson, J. Dell, Class. Quantum. Grav. **8**, 59 (1991)
- [12] M. Raiteri, M. Ferraris, M. Francaviglia, in: “*Gravity, Particles and Space-Time*”, edited by P. Pronin and G. Sardanashvily (World Scientific, Singapore, 1996) p. 81
- [13] M. Ferraris, J. Kijowski, Letters in Math. Phys. **5** 127 (1981); M. Ferraris, M. Francaviglia, C. Reina, Gen. Rel. Grav. **14** (3), 243 (1982); M. Ferraris, J. Kijowski, Rend. Sem. Mat. Univers. Politecn. Torino, **41** (3), 169 (1983)
- [14] L. Fatibene, M. Francaviglia, in; *Seminari di Geometria 1996-1997, Università degli Studi di Bologna, Dipartimento di Matematica*, edited by S. Coen (Tecnoprint, Bologna, 1998) p. 69
- [15] L. Fatibene, M. Ferraris and M. Francaviglia, M. Godina, Gen. Rel. Grav. **30** (9), 1371 (1998)
- [16] I. Kolář, P. W. Michor, J. Slovák, *Natural Operations in Differential Geometry* (Springer–Verlag, New York, 1993)

- [17] L. Fatibene, M. Ferraris, M. Francaviglia, M. Godina, *A geometric definition of Lie derivative for Spinor Fields*, in: Proceedings of “6th International Conference on Differential Geometry and its Applications, August 28–September 1, 1995”, (Brno, Czech Republic), Editor: I. Kolář, (MU University, Brno, Czech Republic, 1996) p. 549
- [18] G. Magnano, M. Ferraris, M. Francaviglia, J. Math. Phys. **31** (2), 378 (1990)
- [19] L. Fatibene, M. Ferraris, M. Francaviglia, M. Raiteri, Ann. Phys. (in press); e-archive: hep-th/9810039
- [20] M. Ferraris, J. Kijowski, Rend. Sem. Mat. Univers. Politecn. Torino **41**, 3 (1983)
- [21] M. Ferraris, J. Kijowski, Gen. Rel. Grav. **14** (2), 165 (1982)
- [22] P. Peldan, Class. Quantum Grav. **9**, 2079 (1992)
- [23] F. A. Lunev, Phys. Lett. **295B**, 99 (1992)
- [24] J.-I. Koga, K.-I. Maeda, *Equivalence of black hole thermodynamics between a generalized theory of gravity and the Einstein theory*, e-print:gr-qc/9803086